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HISTORY OF ZERO AND INFINITY

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INTRODUCTION

0 (**zero**; BrE: /ˈzɪərəʊ/ or AmE: /ˈziːroʊ/) is both a number[1] and the numerical digit used to represent that number in numerals. It fulfils a central role in mathematics as the additive identity of the integers, real numbers, and many other algebraic structures. As a digit, 0 is used as a placeholder in place value systems. In the English language, 0 may be called **zero**,**nought** or (US) **naught** (pron.: /ˈnɔːt/), **nil**, or — in contexts where at least one adjacent digit distinguishes it from the letter "O" — **oh** or **o** (pron.: /ˈoʊ/). Informal or slang terms for zero include **zilch** and **zip**. Ought or aught (pron.: /ˈɔːt/) has also been used historically.

REVIEW OF LITERATURE

Georg Cantor (1845 - 1918) was a student of Dedekind and inherited from him the problem of establishing the class of functions which has a converging Fourier series. Following his teacher, he began to study families of functions having convergence Fourier series as classified by their exceptional points. That is, following even the first ideas of convergence, Cantor expanded the number of exceptional points a function may have and still have a converging Fourier series — except at those points. His first attempt in 1872 allowed for an infinite number of exceptional points answering a question of Riemann.

Here are the details. Given an infinite set of points 6. Define the derived of 6 ,6_3, to be the set of limit points of 6. Define 6_{33} to be the derived set of 6_3, also called the second derived set of 6, and so on. Cantor was able to show that if the trigonometric series converges to zero except at a set of points which has a finite NWK derived set, for some (finite) N, then DQ EQ $_{33}$ Q $_{33}$ $_{33}$ $_3$ $_3$ $_3$. In this paper he also showed the existence of such sets for every Q.

^{3}

$_3$; DQ FRV Q[$_3$ EQ VLQ Q[

Q $_3$

Cantor most certainly was aware that the process of derivations could be carried out indefinitely.

Use the notation 6_3Q_3 to be the QWK derived set of 6. Then 6_3Q_{333} $_36_3Q_{333}$, the derived set of 6Q. Defining in this way 6_{333} to be those points in 6_3Q_3 for every finite Q, we can continue to apply the derive operation. Thus, we get the following sets of points:

3 3 3 $_3$T

333 3 6333 33 3 3 6 3 333 3 3 3 3 6 3 33 3 3 3

6_3 6 3 3 3 6 3 3 3 3 3

3 3

$_3$T 3 3

3 3 3 3 3 3

6 3 33 3 3 3 3 3 6 3 6 3 3 3 3 3 3

The number $_3$ appears naturally in this context. So also do numbers

3 3 33 3 3 3 and so on. The root of these infinite numbers was the attempt to solve a problem of analysis.

However, Cantor now devoted his time to the set theoretic aspects of his new endeavor, abandoning somewhat the underlying Fourier series problems. He first devoted his time to distinguishing the sets of rationals and reals. In 1874, he established that the set of algebraic numbers$_3$ can be put into one-to-one correspondence with the natural numbers. $_3$ but the set of real numbers cannot be put into such a correspondence. We show the simpler

Theorem. The set of rationals is one-to-one correspondence with the natural numbers.

Proof #1. Let UP_3Q ~~P~~ be a rational number represented in reduced

Q

form. Define the relation

$$UP_3Q \;{}_3\,3^P\,3^Q$$

This gives the correspondence of the rationals to a subset of the natural numbers, and hence to the natural numbers.

■

Proof #2. $_3$ (Arrange all the rationals in a table as shown below. Now count the numbers as shown by the arrows. This puts the rationals into correspondence with the natural numbers. As you may note, there is some duplication of the rationals. So, to finish, simply remove the duplicates. Alternatively, build the table with the rationals already in lowest order.

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5 ,
1 , 4 , ...

5 /2
1 , 2 , , ,...

5/3 ,
1 / , , 4/3 , ...

1/4 3/4 5/4 ,
, , , 4/4 , ...

3 /5 5/5 ,
1 , 2 /5 , , 4/ 5 , ...

■

The proof for algebraic numbers is only slightly more complicated.

The proof of the other result, that the real numbers cannot be put into such a correspondence invoked a new and clever argument. Called Cantor's diagonal method, it has been successfully applied to many ends.

Theorem. The set of reals cannot be put into one-to-one correspon-dence with the natural numbers.

First Proof. We give here the 1891 proof. Restrict to the subset of reals in the interval $_{333}$ $_{33}$. Supposing they are denumerable as the set ID_QJ_{3Q} $_3$, we write their decimal expansions as follows:

$_{3\ 3}\ G_{333}\ G_{333}\ G_{333}$
D_3 $_{33\ 3}$

$_{3\ 3}\ G_{333}\ G_{333}\ G_{333}$
D_3 $_{33\ 3}$

$_{3\ 3}\ G_{333}\ G_{333}\ G_{333}$
D_3 $_{33\ 3}$

.

.

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where the G's are digits 0 - 9. Now define the number

D 3 3 G3 G3 G3 3 3 3

by selecting G3 3 G3333 G3 3 G3333 G3 3 G3333 3 3 3 3 This gives a numbernot

in the set IDQJ3, and the result is proved.

Q 3

■

Second Proof. This proof, which appeared in 1874, is not as well known. We show that for any sequence Y33 Y33 33 3 of reals there is a number that is not in the sequence in any interval of real numbers 3D3 E3. First, let D3 and E3 be the first members of the sequence in 3D3 E3 with D3 3 E3. Let D3 and E3 be the first members of the sequence in 3D33 E33 with D3 3 E3, and so on. Thus, D33 D33 3 3 3 is an increasing sequence, and E33 E33 3 33 is a decreasing sequence. There are three cases. If the sequences are finite, then any number inside the last chosen interval satisfies the requirement. Suppose now the sequences are infinite and they converge to limits, D3 and E3, respectively. If they are equal, then this value satisfies the requirement. If not, any value in the open interval 3D33 E33 does so.

■

Seeking undenumerable sets, Cantor considered topological notions for

his derived sets. We say a set 6 Ÿ 3D3 E3 is dense if 3D3 E3.
63 We

say 6 is closed if 63? 6 63. We say 6 is isolated if 63 3. Finally,

we say 6 is perfect if 63 6. Remarkably, Cantor showed that perfect

sets must be uncountable. One of the most famous perfect sets is so-called the middle thirds set defined as the residual of the open interval

333 33 by first removing the middle third (i.e., 3 3 3 3). Next remove
3 the

3 3

middle thirds of the two subintervals remaining and the middle thirdsof the four remaining subintervals after that, and so on. This set is one of the first examples of an uncountable Lebesgue measurable set of measure zero that mathematics graduate students learn.

At this point he was in possession of two orders of infinity, countable and uncountable infinity. Being unable to determine an infinity in be-tween, he gave a proof that every set of points on the line could be put in one-to-one correspondence with either the natural numbers or reals. His proof was incorrect, but his quest is known today and is called the continuum hypothesis. The problem is open today and is complicated. In 1938, Kurt Godel¨ proved that the continuum hypothesis cannot be disproved on the basis of the set-theoretic principles we accept today. Moreover, in 1963, Paul Cohen established that it cannot be proved within these principles. This means that the continuum is undecidable.

Cantor was not without detractors. Though his methods were enthusi-astically received by some mathematicians, his former teacher Leopold Kronecker believed that all of mathematics should be based on the natural numbers. This may be called finitism. He also believed that mathematics should be constructed, and this is called constructivism. He soundly rejected Cantor's new methods, privately and publicly. As a journal editor, Kronecker may have delayed the publication of Cantor's work.

By 1879 Cantor was in possession of powers of infinity, defining two sets to be of the same power if they can be placed into one-to-one correspondence. Using his diagonalization method, he was able to demonstrate orders or powers of infinity of every order. Here is how to exhibit a set of higher power than that of the reals. Let) be the set of real-valued functions defined on the reals. Assume that this class of functions has the same power as the reals. Then they can be counted as $I_{\backslash 3}[_3$, where both [and \ range over the reals. Define a new function $I\ _3[_3$ such that

$$I\ _3[_{3\ _3}\ I[_3[_3$$

for each real [. This function cannot be in the original set). In turn, this method can be applied recursively to obtain higher and higher powers of infinity. There is another connection with subsets of sets. Indeed, in the argument above the subset of) consisting of functions assuming only the values 0 and 1 could have been used. In such a way it is possible to see that we are looking at the set of all subsets of the reals. A subset corresponding to a particular function is the set of values for which it has the value 1. Conversely, any subset generates a function according to the same rule.

In all this, infinity is now a number in its own right, though it is linked with counting ideas and relations to sets of sets. The term power gave us the expression power set or set of subsets of a given set. For a finite set with Q elements, the set of all subsets has size $_3Q$. However, the power of a set is an attribute of a set akin to the cardinality of a set. Two sets have the same power if they can be put in one-to-one correspondence.

In about 1882, Cantor introduced a new infinity, distinguishing car-dinality from order, cardinal numbers from ordinal numbers. (i.e., one, two three from first, second, third). He would say that $_3D_{33}$ D_{33} $_3$ $_3$ $_{33}$ and $_3E_{33}$ E_{33} $_3$ $_3$ $_3$ $_3$ E_{33} have the same cardinality or power, but their order is different. The first has order $_3$ while the second has order $_3$ $_3$ $_3$. For finite sets, there is only one order that can be given, even though elements can be transposed. Therefore, ordinal and cardinal numbers can be identified.

Using a method similar to the second proof above, Cantor showed how to produce a set with power greater than the natural numbers, namely, the set of all ordinal numbers of the power of the natural numbers. From this, he went on to construct the power set of the set of ordinals, and so on generating higher and higher powers. Now, to make contact with the power of the real numbers, Cantor made the assumption that the reals were well-ordered, which is defined below. From this, he established that the power (cardinality) of the real numbers is less than, equal to or greater than each of the new powers, but not which of them it is.

Notation: By 1895 Cantor defined cardinal exponentiation. Using the term $\#_3$ (aleph-null) to denote the cardinality of the natural numbers, he defined $_{333}$ for the cardinality of the reals. With $\#_3$ (and more generally $\#_3$ denoting the $_3$ WK cardinal) the next larger cardinal than

$\#_3$, the continuum hypothesis is written as $_3$ $_3$ $_3$ $\#_3$.

Cantor and others produced similar examples of a special category of nowhere-dense sets as an application arose of these ideas. First, a nowhere dense set 6 is a set for which the complement of its closure

is dense, i.e. a6› is dense. The set of binary fractions I $_3$ J$_3$ and

$\overline{_3}\,{}^3\ {}_3$

the Cantor middle thirds set are nowhere dense, but the rationals are dense. The special new category consists of those that are “fat” in the following way: Every finite covering of the set by intervals should have total length greater than some given number, say 1. It becomes natural to say that such sets have content, and the content of the particular nowhere dense set under consideration is the infimum of the total length of all finite coverings. The idea of content was to play a major role in the development of the modern integral, notably the Jordan completion to the Riemann-Cauchy integral and ultimately the Lebesgue integral. So, we see here, sets and infinity now giving rise to new ideas for analysis. And note that the Fourier series problem that served as the root of these investigations would find its ultimate solution within the context of the modern integral. At this point we have come full circle. The problem created the solution. In 1873, the French mathematician Paul du Bois Reymond (1831 - 1889) discovered a continuous function for which its Fourier series diverged at a single point, solving a long-standing open problem. That this was the tip of the iceberg on divergence of Fourier series is illustrated below by three theorems. These results are essentially the current best possible pointwise results for Fourier series. We first need the definition: A set (Ÿ 5 is said to have measure zero if for every”! ₃ there exists a finite set of intervals, ₃ ₃, ₃ ₃ ₃ ₃ ₃ ₃, N on for which

1. (Ÿ >N ,L

L ₃

2. 3N M,LM, where for any interval ,, M,M is the length of ,.

L ₃

(Of course, N and the intervals, ₃ ₃, ₃ ₃ ₃ ₃ ₃ ₃, N depend on the set (and on ”.)

CONCLUSION

In some ways, the paradoxes and overall lack of agreement on basic principles in set theory can be seen as parallel to the paradoxes and overall lack of agreement on basic principles in the early days of calculus or noneuclidean geometry. Parallel to that, no doubt there were many paradoxes and overall lack of agreement of basic principles in the fledgling subject of geometry more than two thousand years earlier. It seems that by making various decisions about infinity via its “agents”, the axiom of choice and the well ordering axiom, different systems of mathematics result. Therefore, the original absolute axiomatic model of Euclidean geometry within which all propositions can be resolved and that all of science has tried to emulate, is gone forever. Infinity and these trappings of set theory so very much needed to advance the early and modern mathematical theories, has served up a second dish, the demiseof certainty.

Will the issues of infinity ever be resolved to the satisfaction of logicians and mathematicians? Like the limit, the understanding of which was finally assimilated after two millenia, a working definition of infinity satisfactory to all practitioners will probably percolate out. For most of us that point has already been achieved.

REFERENCES

Bemer, R. W. (1967). "Towards standards for handwritten zero and oh: much ado about nothing (and a letter), or a partial dossier on distinguishing between handwritten zero and oh".Communications of the ACM ***10*** *(8): 513–518.doi:10.1145/363534.363563.*

Bunt, Lucas Nicolaas Hendrik; Jones, Phillip S.; Bedient, Jack D. (1988). The historical roots of elementary mathematics. Courier Dover Publications. pp. 254–255. ISBN 0-486-2556- 3., Extract of pages 254-255

Catherine Soanes, ed. (2001). The Oxford Dictionary, Thesaurus and Word power Guide (Hardback). Maurice Waite, Sara Hawker (2nd ed.). New York, United States: Oxford University Press. ISBN 978-0-19-860373-3.

Devlin, Keith, Mathematics: The Science of Patterns: The Search for Order in Life, Mind and the Universe (Scientific American Paperback Library) 1996, ISBN 9780716750475 ^ The Oxford Dictionary of English Etymology, Oxford English Dictionary, sub "mathematics", "mathematic", "mathematics"

Ifrah, Georges (2000). The Universal History of Numbers: From Prehistory to the Invention of the Computer. Wiley. ISBN 0-471-39340-1.

Kang-Shen Shen; John N. Crossley; Anthony W. C. Lun; Hui Liu (1999). The Nine Chapterson the Mathematical Art: Companion and Commentary. Oxford University Press. p. 35.ISBN 978-0-19-853936-0. "Zero was regarded as a number in India... whereas the Chinese employed a vacant position"

Luke Hodgkin (2 June 2005). A History of Mathematics: From Mesopotamia to Modernity:From Mesopotamia to Modernity. Oxford University Press. p. 85. ISBN 978-0-19-152383-0.

Menninger, Karl (1992). Number words and number symbols: a cultural history of numbers.Courier Dover Publications. p. 401. ISBN 0-486-27096-3.

Russel, Bertrand (1942). Principles of mathematics (2 ed.). Forgotten Books. p. 125. ISBN 1-4400-5416-9., Chapter 14, page 125

S. Dehaene; G. Dehaene-Lambertz; L. Cohen (Aug 1998). "Abstract representations of numbers in the animal and human brain". Trends in Neuroscience ***21*** *(8): pp. 355-361. doi:10.1016/S0166-2236(98)01263-6.*